Linear Functionals

(Section 8.1)

Definition: Let V be a vector space over $\mathbb{F}$. A *linear functional* on V is a linear map $f: V \to \mathbb{F}$.

Example 1: $(\mathbb{R}^n)$

Define $f : \mathbb{R}^n \to \mathbb{R}$

$$\boxed{f\left((a_i)_{i=1}^n\right) = \sum_{i=1}^n a_i \in \mathbb{R}}$$

Since $a_i \in \mathbb{R}\ \forall\ 1 \leq i \leq n$

Why is f linear?

Let $\alpha \in \mathbb{R}$, $(b_i)_{i=1}^n \in \mathbb{R}^n$

$$f\left(\alpha (a_i)_{i=1}^n + (b_i)_{i=1}^n\right) = ?$$

$$f\left(\alpha(a_i)_{i=1}^{n}+(b_i)_{i=1}^{n}\right)=$$

$$=f\left((\alpha a_i)_{i=1}^{n}+(b_i)_{i=1}^{n}\right)$$

$$=f\left((\alpha a_i+b_i)_{i=1}^{n}\right)$$

$$=\sum_{i=1}^{n}(\alpha a_i+b_i)$$

$$=\sum_{i=1}^{n}\alpha a_i+\sum_{i=1}^{n}b_i$$

$$=\alpha\sum_{i=1}^{n}a_i+\sum_{i=1}^{n}b_i=\alpha f\left((a_i)_{i=1}^{n}\right)+f\left((b_i)_{i=1}^{n}\right)\checkmark$$

Example 2: $(M_n(\mathbb{R}))$

Define $Tr : M_n(\mathbb{R}) \to \mathbb{R}$

if $A = (A_{i,j})_{i,j=1}^{n} \in M_n(\mathbb{R})$,

$$\boxed{Tr(A) = \sum_{i=1}^{n} A_{i,i}},$$

the **Trace** of A,

is a linear functional.

Example 3: $(\mathcal{S}(\mathbb{R}))$

$$f : \mathcal{S}(\mathbb{R}) \to \mathbb{R}$$

$$\boxed{f\left((a_i)_{i=1}^{\infty}\right) = a_{21} \in \mathbb{R}}$$

Then f is a linear functional

Definition: (dual space) Let V be a vector space over $\mathbb{F}$. Then the dual space of V is the collection of all linear functionals $f : V \to \mathbb{F}$.

Recall: with the operations

$$(f+g)(x) = f(x) + g(x)$$
$$(\alpha \cdot f)(x) = \alpha \cdot f(x) \quad \forall$$

$\alpha \in \mathbb{F}$, $x \in V$, f, g in the dual, the dual space is a vector space.

Notation: For a vector space V over $\mathbb{F}$, V^* will denote the dual space to V.

Theorem: (finite dimensions)

Let V be a finite dimensional vector space over $\mathbb{F}$. Then V^* is isomorphic to V as a vector space. As a consequence,

$$\dim_{\mathbb{F}}(V) = \dim_{\mathbb{F}}(V^*)$$

Proof: Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$

be a basis for V.

Define the dual basis

$$\{v_1^*, v_2^*, \ldots, v_n^*\}$$

as follows: if $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$

and $x = \sum_{i=1}^{n} \alpha_i v_i$, then

$$\boxed{v_k^*(x) = \alpha_k}$$

Define $\varphi: V \to V^*$ by

$$\varphi\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i v_i^*$$

$\alpha_1, \ldots, \alpha_n \in \mathbb{F}$

Since $\mathcal{B}$ is a basis, this defines φ unambiguously on V (i.e. φ is well-defined).

Show: 1) φ is linear

2) φ is bijective

1) Let $x = \sum_{i=1}^{n} \alpha_i v_i$, $y = \sum_{i=1}^{n} \beta_i v_i \in V$

for $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{F}$

and let $c \in \mathbb{F}$.

$$\varphi(cx + y)$$
$$= \varphi\left(c \sum_{i=1}^{n} \alpha_i v_i + \sum_{i=1}^{n} \beta_i v_i\right)$$

$$= \varphi\left(\sum_{i=1}^{n} (c\alpha_i + \beta_i) v_i\right)$$

$$= \sum_{i=1}^{n} (c\alpha_i + \beta_i) v_i^*$$

$$= \sum_{i=1}^{n} (c\alpha_i) v_i^* + \sum_{i=1}^{n} \beta_i v_i^*$$

$$= c\sum_{i=1}^{n} \alpha_i v_i^* + \sum_{i=1}^{n} \beta_i v_i^*$$

true since V^* is a vector space over $\mathbb{F}$

$$= c\,\varphi(x) + \varphi(y)$$

2) a) φ injective.

Since we know φ is linear, injectivity is equivalent to

$$\varphi(x) = 0_{V^*} \Rightarrow x = 0_V.$$

Write $x = \sum_{i=1}^{n} \alpha_i v_i$ and suppose

$\varphi(x) = 0_{V^*}$. Then

$$\varphi(x)(y) = 0_{\mathbb{F}} \quad \forall \quad y \in V$$

But with $y = v_j$, $1 \le j \le n$,

$$\varphi(x)(v_j)$$

$$= \varphi\left(\sum_{i=1}^{n} \alpha_i v_i\right)(v_j)$$

$$= \sum_{i=1}^{n} \alpha_i v_i^*(v_j)$$

$= \alpha_j$ since

$$v_i^*(v_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

But we assume

$$\varphi(x)(y) = 0_{\mathbb{F}} \quad \forall\, y \in V$$

$$\Rightarrow \varphi(x)(v_j) = 0_{\mathbb{F}} \quad \forall\, 1 \leq j \leq n$$

$$\Rightarrow \alpha_j = 0_{\mathbb{F}} \quad \forall\, 1 \leq j \leq n$$

$$\Rightarrow x = 0_V. \quad \checkmark$$

ii) φ is surjective

Let $f \in V^*$. Recall f is linear,

so

$$f\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i f(v_i)$$

Let $\beta_j = f(v_j) \in \mathbb{F}$ and

define $g = \sum_{j=1}^{n} \beta_j v_j^*$

goal: Show $g = f$.

$$g\left(\sum_{i=1}^{n}\alpha_i v_i\right)$$

$$= \sum_{i=1}^{n}\alpha_i\, g(v_i) \quad \text{(linearity)}$$

$$= \sum_{i=1}^{n}\alpha_i\left(\sum_{j=1}^{n}\beta_j \underbrace{v_j^*(v_i)}_{\delta_{i,j}}\right)$$

$$= \sum_{i=1}^{n}\alpha_i\beta_i$$

$$= \sum_{i=1}^{n}\alpha_i f(v_i) = f\left(\sum_{i=1}^{n}\alpha_i v_i\right) \checkmark$$

This shows $f = g$, and

so $\varphi\left(\sum_{j=1}^{n} \beta_j v_j\right) = \sum_{j=1}^{n} \beta_j v_j^*$

$$= g$$

$$= f$$

and so, φ is surjective.

Hence, we may conclude that

φ is an isomorphism from

V to V^*! $\square$

Observation: The proof of this theorem does not generalize to infinite dimensions. If $V = \mathbb{P}[x, \mathbb{R}]$, $\{x^i\}_{i=0}^{\infty}$ is the monomial basis, then if we set

$$\varphi\left(\sum_{i=0}^{n} \alpha_i x^i\right) = \sum_{i=0}^{n} \alpha_i ,$$

then φ is not in the span of the dual basis to $\{x^i\}_{i=1}^{\infty}$

When working in infinite-dimensions, V^* is called the algebraic dual and is usually replaced by a subset of V^*, which we then refer to as the dual.